A SECOND COURSE IN GENERAL TOPOLOGY

Next we show that continuity of mappings between spaces can be very naturally characterized in terms of filter convergence. The intuitive idea of continuity of a map f from X to Y is that f does not "tear" the space mapping, and this is equivalent with the idea that for points X the images are "near r to each

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Introduction

Metric spaces and compact Hausdorff spaces are the most important and useful of the various kinds of spaces considered in general topology. In these notes, we deal with these two kinds of spaces, and also with paracompact spaces. One of the main themes in these notes is the construction and the use of continuous pseudometrics. This is connected with two central areas of general topology: covering properties and metrizability. The connection to covering properties results from the "Stone Coincidence Theorem" (here *Stone* does *not* refer to M.H. Stone, famous from *Stone-Čech compactifications* ja *Stone-spaces*, but rather to A.H. Stone, who is perhaps best known for his *flexagons*). Stone proved the coincidence theorem in 1948, and it states the equivalence of two covering properties, "full normality", defined by J.W. Tukey in 1940, and "paracompactness", defined by J. Dieudonne in 1944. Full normality was forgotten after Stone's theorem (and Tukey became later famous as a pioneer of *data mining*), but results of Tukey and Stone implied one of the fundamental results of "modern general topology": *every (pseudo)metric space is paracompact.* This result makes it possible to use pseudometrics in the theory of paracompact spaces and to use paracompactness to study metrizability. One of the techniques used in these studies is that of *partitions of unity.*

Other topics considered in these notes include Stone-Čech compactifications, continuous selections function spaces and approximation of real and complex functions on compact spaces.

I used an early version of these lecture notes for a second course in general topology which I gave at the University of Helsinki in 2007. In the lectures, I also dealt with the construction of spaces by means of infinitary combinatorics, but the material on that topic is not included here.

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Conventions

In the following, a *space* means a topological space (unless specified otherwise). We usually denote a space by a symbol like X instead of writing, say, (X, τ), and we sometimes denote the topology of a space X by τ_X.

We write $A \overset{c}{\subset} X$ to indicate that A is a closed subset of X, and we write $A \overset{o}{\subset} X$ to indicate that A is an open subset of X. In particular, we have that $\tau_X = \{G : G \overset{o}{\subset} X\}$.

For a set $A \subset X$, we denote by $\overline{A}$ or by $\operatorname{Cl} A$ the closure of the set A in the space X. Sometimes we use the more precise notation $\overline{A}^X$ or $\operatorname{Cl}_X A$. The symbols $\operatorname{Int} A$ and $\operatorname{Int}_X A$ denote the interior of A in X.

A *neighbourhood* (abreviated "nbhd") of a set $A \subset X$ in the space X is a set $E \subset X$ such that $A \subset \operatorname{Int} E$. If $A = \{x\}$, we speak of a neighbourhood of the point x in the space X. Note that here neighbourhoods are *not* necessarily open sets (unlike in some text-books).

A *neighbourhood base* of a set $A \subset X$ (or of a point $x \in X$) is a family $\mathcal{B}$ of nbhds of A (of x) such that every nbhd of A (of x) contains some set of the family $\mathcal{B}$. A *closed (or open, clopen, etc) neighbourhood base* is a nbhd base consisting of closed (open, clopen, etc) sets.

For a subset A of a space X, we denote by η_A, or by $\eta_A(X)$, the *neighbourhood filter* $\{E \subset X : A \subset \operatorname{Int}(E)\}$ of the set A. If $A = \{x\}$, we write η_x and $\eta_x(X)$ in room of η_A and $\eta_A(X)$, respectively, and we speak of the *neighbourhood filter of the point* x.

For a product space $X = \prod_{i \in I} X_i$, we denote by p_j the *projection* $(x_i)_{i \in I} \mapsto x_j$ of the product onto its *jth factor* X_j.

The symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{P}$ and $\mathbb{I}$ denote the sets consisting of all real numbers, all rational numbers, all irrational numbers and $\{x \in \mathbb{R} : 0 \leq x \leq 1\}$, respectively. We use the same symbols to denote the corresponding *Euclidean spaces*.

We let $\mathbb{N} = \{1, 2, 3, 4, ...\}$ and $\omega = \{0, 1, 2, 3, 4, ...\}$.

"TFEA" is an abbreviation for the phrase "the following are equivalent".

I COMPLETE REGULARITY

I.1. Definition and basic properties

Recall that a (topological) space X is

– T_0 if $\forall\, x, y \in X$, if $x \neq y$, then $\exists\, U \in \eta_x$ with $y \notin U$ or $\exists\, V \in \eta_y$ with $x \notin V$.

– T_1 if $\forall\, x, y \in X$, if $x \neq y$, then $\exists\, U \in \eta_x$ with $y \notin U$ and $\exists\, V \in \eta_y$ with $x \notin V$.

Note that X is T_1 iff $\{x\} \Subset X$ for every $x \in X$.

– T_2 if $\forall\, x, y \in X$, if $x \neq y$, then $\exists\, U \in \eta_x$ and $\exists\, V \in \eta_y$ with $U \cap V = \emptyset$.

(T_2-spaces are also called *Hausdorff spaces.*)

– *regular* if $\forall\, F \Subset X$ and $\forall\, x \in X \setminus F$, there $\exists\, U \in \eta_x$ and $\exists\, V \in \eta_F$ with $U \cap V = \emptyset$.

Note that a space is regular iff every point has a closed neighbourhood base.

– T_3 if X is regular and T_1.

– *normal* if $\forall\, F, S \Subset X$, if $F \cap S = \emptyset$, then $\exists\, U \in \eta_F$ and $\exists\, V \in \eta_S$ with $U \cap V = \emptyset$.

Note that a space is normal iff every closed subset has a closed neighbourhood base.

– T_4 if X is normal and T_1.

Note that we have

$$\boxed{T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0}$$

Definition A topological space X is *completely regular* if for all $F \Subset X$ and $x \in X \setminus F$, there exists a continuous $f : X \to \mathbb{I}$ such that $f(x) = 0$ and $f(F) \subset \{1\}$.
The space X is a *Tihonov space* if X is completely regular and T_1.

(Tihonov spaces are sometimes called $T_{3\frac{1}{2}}$-spaces.)

Note that completely regular⇒regular. Moreover, the Urysohn Lemma shows that T_4⇒Tihonov.

It is well known and easy to see that regularity and the properties T_2, T_1 and T_0 are both hereditary and productive. The same is true for complete regularity.

1 Proposition *(i) Every subspace of a completely regular space is completely regular. (ii) Any product of completely regular spaces is completely regular.*

Proof. Exercise. □

Remember that normality is not, in general, preserved either with passing to subspaces or in forming products (even in the case of only two factors).

Completely regular spaces have "enough continuous real-valued functions".

Recall that when X is a set, Y is a topological space and F is a set of mappings $X \to Y$, then the *weak topology* on X determined by F is the coarsest topology on X which makes each $f \in F$ continuous, i.e., the topology on X which has the family $\{f^{-1}(O) : f \in F \text{ and } O \Subset Y\}$ as a subspace.

2 Lemma *X is completely regular iff X has the weak topology determined by a set of functions $X \to \mathbb{I}$.*

Proof. Exercise. □

For spaces X and Y, we denote by $C(X, Y)$ the set of all continuous mappings $X \to Y$. We write $C(X)$ for $C(X, \mathbb{R})$.

We show next that Tihonov spaces coincide with spaces which can be embedded in "cubes". By a *cube* we mean a product space of the form $\mathbb{I}^A$, where A is any set (recall that $\mathbb{I}^A = \{f : f \text{ is a mapping } A \to \mathbb{I}\}$, in other words, $\mathbb{I}^A = \{(r_a)_{a\in A} : r_a \in \mathbb{I} \text{ for every } a \in A\}$, in other words, $\mathbb{I}^A = \prod_{a\in A} \mathbb{I}_a$, where $\mathbb{I}_a = \mathbb{I}$ for every $a \in A$; the topology on $\mathbb{I}^A$ is the usual product topology).

Recall that a mapping $\phi : X \to Y$ is an *embedding* of the space X in the space Y if ϕ is a homeomorphism between X and the subspace $\phi(X)$ of Y.

3 Theorem *Let X be a Tihonov space. Denote the set $C(X, \mathbb{I})$ by F, and define a mapping $\varphi : X \to \mathbb{I}^F$ by the condition $\varphi(x)_f = f(x)$.*
A. The mapping φ is an embedding of X into $\mathbb{I}^F$.
B. Every continuous function $\varphi(X) \to \mathbb{I}$ can be extended to a continuous function $\mathbb{I}^F \to \mathbb{I}$.

Proof. A. Since X is completely regular and T_1, we see that, for all $x, y \in X$, if $x \neq y$, then there exists $f \in F$ such that $f(x) = 0$ and $f(y) = 1$. It follows that the mapping φ is one-to-one. The mapping φ is continuous, because for every $f \in F$, the composition

$p_f \circ \varphi$ is the same as the (continuous) mapping f: by the definition of φ, we have, for every $x \in X$, that $(p_f \circ \varphi)(x) = p_f(\varphi(x)) = \varphi(x)_f = f(x)$.

By Lemma 2, the family $\mathcal{E} = \{f^{-1}(O) : f \in F \text{ and } O \Subset \mathbb{I}\}$ is a subbase of X. To show that φ is an open mapping $X \to \varphi(X)$, it suffices (since φ is $1-1$) to show that $\varphi(E) \Subset \varphi(X)$ for every $E \in \mathcal{E}$. Let $h \in F$ and $O \Subset \mathbb{I}$. Then we have that

$$\begin{aligned}\varphi(h^{-1}(O)) &= \{\varphi(x) : x \in X \text{ and } h(x) \in O\} \\ &= \{\varphi(x) : x \in X \text{ and } \varphi(x)_h \in O\} \\ &= \{(r_f)_{f\in F} \in \varphi(X) : r_h \in O\}\,. \\ &= \varphi(X) \cap \{(r_f)_{f\in F} \in \mathbb{I}^F : r_h \in O\}\,.\end{aligned}$$

We have shown that the set $\varphi(h^{-1}(O))$ is the intersection with $\varphi(X)$ of a standard subbasic set of the product space $\mathbb{I}^F$; hence $\varphi(h^{-1}(O)) \Subset \varphi(X)$. We have shown that φ is an open mapping. Hence φ is an embedding.

B. Let g be a continuous function $\varphi(X) \to \mathbb{I}$. Then the function $h = g \circ \varphi$ is continuous $X \to \mathbb{I}$. For every $x \in X$, we have that $g(\varphi(x)) = h(x) = \varphi(x)_h = p_h(\varphi(x))$. As a consequence, we have that $g = p_h|\varphi(X)$ and we can set $\bar{g} = p_h$ to obtain a continuous extension of g which is defined on $\mathbb{I}^F$. □

Part A of Theorem 3 has the following important consequence.

4 Corollary *(i) A space is a Tihonov space iff the space is homeomorphic with a subspace of some cube.*
(ii) A space is a compact Hausdorff space iff the space is homeomorphic with a closed subspace of some cube.

Proof. (i) *Necessity* follows from Theorem 3.
Sufficiency. Since $\mathbb{I}$ is a Tihonov space, it follows from Proposition 1 that any subspace of a cube is Tihonov.

(ii) *Necessity.* A compact Hausdorff space K is normal and hence Tihonov. By (i), X is homeomorphic with a subspace of some cube; moreover, the cube is a Hausdorff space, and hence any compact subspace of it is closed.
Sufficiency. By the Tihonov Theorem (see Section II.3), every cube is compact; hence every closed subspace of a cube is compact. □

Note that when $\varphi : X \to Y$ is an embedding, we can construct an *extension* of X which is homeomorphic with Y, in other words, we can represent X as a subspace of a space Z, which is a homeomorphic copy of Y. To obtain such a space, we write $Z = (Y \smallsetminus \varphi(X)) \cup X$ (here we are assuming that $Y \smallsetminus \varphi(X)$ and X have no common points; if this is not the case, then we must first "make $Y \smallsetminus \varphi(X)$ disjoint from X" before we form the union). We declare a set $G \subset Z$ to be open if there exists $U \Subset Y$ such that $G = (U \smallsetminus \varphi(X)) \cup \varphi^{-1}(U)$. Now the space Z is homeomorphic with Y and the space X is a subspace of Z.

The preceding observation and Corollary 4 have the following consequence.

5 Corollary *Tihonov spaces coincide with subspaces of compact Hausdorff spaces.*

A *compactification* of a space X is a compact space Z such that X is a dense subspace of Z. If K is any compact space having X as a subspace, then the subspace $\overline{X}$ of K is a compactification of X.

We can restate the preceding corollary as follows.

5' Corollary *A space X is a Tihonov space iff X has a Hausdorff compactification.*

This result gives a characterization for Tihonov spaces which is *purely topological* in the sense that it (unlike the definition) does not involve the set $\mathbb{R}$ with its special properties. However, it is an "external" characterization depending on other spaces. In Theorem II.4.14 below, we shall give a purely topological "internal" characterization for Tihonov spaces.

We shall also derive some consequences of part B of Theorem 3. First we note that the conclusion of part B can be significantly strengthened.

Let Z and Y be topological spaces, and let X be a subspace of Z. We say that X is Y*-embedded* in Z provided that every continuous mapping $f : X \to Y$ can be extended to a continuous mapping $\bar{f} : Z \to Y$.

6 Lemma *Assume that X is Y-embedded in Z. Then, for every set A, X is Y^A-embedded in Z.*

Proof. Let A be a set, and let f be a continuous mapping $X \to Y^A$. For every $a \in A$, the composition $p_a \circ f$ is a continuous mapping $X \to Y$ and it follows, since X is Y-embedded in Z, that there exists a continuous mapping $g_a : Z \to Y$ such that $g|X = p_a \circ f$. We define a mapping $g : Z \to Y^A$ by the rule $g(z)_a = g_a(z)$, and we note that g is continuous.

Moreover, for each $x \in X$, we have that $g(x)_a = g_a(x) = p_a(f(x)) = f(x)_a$ for every $a \in A$, and hence we have that $g(x) = f(x)$. □

7 Lemma *Assume that X is Y-embedded in Z and X is dense in Z. Then X is F-embedded in Z for every closed subspace F of Y.*

Proof. Let $F \Subset Y$, and let $f : X \to F$ be continuous. Then f is continuous $X \to Y$ and it follows, since X is Y-embedded in Z, that there exists a continuous $g : Z \to Y$ such that $g|X = f$. By continuity of g, we have that

$$g(Z) = g(\overline{X}) \subset \overline{g(X)} = \overline{f(X)} \subset \overline{F} = F\,.$$

Hence g is a mapping $Z \to F$. □

Now we can show that every Tihonov space has a very special compactification.

8 Theorem *Let X be a Tihonov space. Then X has a Hausdorff compactification C such that, for every compact Hausdorff space K, every continuous mapping $X \to K$ can be extended to a continuous mapping $C \to K$.*

Proof. Let φ be as in Theorem 3. Then X has a compactification C homeomorphic with $\overline{\varphi(X)}$, and Part B of Theorem 3 shows that X is $\mathbb{I}$-embedded in C. By Lemma 6, X is Y-embedded in C, for every cube Y. It follows by Lemma 7 that X is L-embedded in C for every closed subspace L of a cube. As a consequence, X is K-embedded in C for every space K which is homeomorphic with a closed subspace of a cube. By Corollary 4, X is K-embedded in C for every compact Hausdorff space K. □

Later we shall see that a compactification C of a Tihonov space X as above is unique (up to a homeomorphism). It is called the *Čech-Stone compactification* of X.

I.2. Some examples

We have mentioned above that T_4⇒Tihonov⇒T_3. In this section, we show that neither of the arrows can be reversed.

Recall that a subset of a topological space is *clopen* if the set is both open and closed. Note that $A \subset X$ is clopen iff the *characteristic function* χ_A of A (i.e., the function $X \to \mathbb{I}$ which has constant value 1 in the set A and constant value 0 in the set $X \setminus A$) is continuous.

A space is *zero-dimensional* if the clopen subsets of the space form a base for the topology of the space, in other words, if every point of the space has a clopen nbhd base. By the previous characterization of clopenness it is clear that *every zero-dimensional space is completely regular.*

The basic open sets $[a, b)$ ($a, b \in \mathbb{R}$ and $a < b$) of the *Sorgenfrey line* S are clopen, and hence S is zero-dimensional. The topology of S is finer than the Euclidean topology of $\mathbb{R}$ and hence S is T_1. As a consequence, S is a Tihonov space.

The space S is a well-known example of a normal space whose square is not normal. Using this result and Proposition 1.1, we have the following result.

1 Example The space S^2 is Tihonov but not normal.

To obtain a regular non-completely regular space, we cannot rely on any previously known results.

2 Example There exists a regular space X which is not non-completely regular.

Proof. We topologize the set $Z = \mathbb{R} \times \mathbb{I}$ as follows: points of the set $\mathbb{R} \times (0, 1]$ are isolated; a point of the form $(r, 0)$ has a nbhd base by sets of the form $V_r \setminus F$, where $V_r = \{(a, b) \in Z : b = |a - r|\}$ and F is a finite subset of the set $V_r \setminus \{(r, 0)\}$.

We topologize the set $X = Z \cup \{-\infty, +\infty\}$ so that Z is an open subspace of X, the point $-\infty$ has a nbhd base by the sets $O_n = \{-\infty\} \cup \{(a, b) \in Z : a \leq -n\}$, for $n \in \mathbb{N}$, and the point $+\infty$ has a nbhd base by the sets $U_n = \{+\infty\} \cup \{(a, b) \in Z : a \geq n\}$, for $n \in \mathbb{N}$.

To see that the space X is regular, note first that the basic nbhds given for points of Z are clopen in X. Moreover, it is easy to see that, for every $n \in \mathbb{N}$, we have $\mathrm{Cl}_X(O_{n+2}) \subset O_n$ and $\mathrm{Cl}_X(U_{n+2}) \subset U_n$. It follows that X is regular.

It remains to show that X is not completely regular. We start by proving the following

Claim. Let F be a closed G_δ-subset of X, and let $E = \{r \in \mathbb{R} : (r, 0) \in F\}$. Assume that $n \in \mathbb{Z}$ and the set $[n - 1, n] \setminus E$ is countable. Then the set $[n, n + 1] \setminus E$ is countable.

Proof of Claim. Since F is a G_δ-set, we can find, for every point $(r, 0) \in F$, a countable set $L_r \subset V_r$ such that $V_r \setminus L_r \subset F$. Since the set $[n-1, n] \setminus E$ is countable, we can choose a countably infinite set $A \subset [n - 1, n] \cap E$. Denote by L the countable set $\bigcup_{a \in A} L_a$, and note that the set $H = \{s \in [n, n + 1] : V_s \cap L \neq \emptyset\}$ is countable. To complete the proof of the claim, we show that the set $J = [n, n+1] \setminus H$ is contained in E. Let $r \in J$. Note that

$V_r \cap (V_a \setminus L_a) \neq \emptyset$ for every $a \in A$. It follows, since A is infinite and since $V_a \setminus L_a \subset F$ for every $a \in A$, that we have $(r, 0) \in \operatorname{Cl} F = F$. As a consequence, $r \in E$. $\square$

Now we use the Claim to show that we have $f(-\infty) = f(+\infty)$ for every $f \in C(X)$. Let $f \in C(X)$. We show that we have $|f(-\infty) - f(+\infty)| \leq \epsilon$ for every $\epsilon > 0$. Let $\epsilon > 0$. By continuity of f, the set $F = \{p \in X : |f(p) - f(-\infty)| \leq \epsilon\}$ is a closed G_δ-set, and there exists $n \in \mathbb{N}$ such that $O_n \subset F$. Let $E = \{r \in \mathbb{R} : (r, 0) \in F\}$. Since $O_n \subset E$, we have that $\{r \in \mathbb{R} : r \leq -n\} \subset E$. Using the Claim and induction, we get that, for every $k \in \mathbb{N}$, the set $[-n - 1 + k, -n + k] \setminus E$ is countable. It follows that the set $\mathbb{R} \setminus E$ is countable. In other words, $(r, 0) \in F$ for all but countably many $r \in \mathbb{R}$. As a consequence, we have that $-\infty \in \operatorname{Cl} F = F$ and hence that $|f(-\infty) - f(+\infty)| \leq \epsilon$.

We have shown that $f(-\infty) = f(+\infty)$ for every $f \in C(X)$. It follows that the space X is not completely regular. $\square$

I.3. Exercises

1. Prove Proposition 1.1.

2. Prove Lemma 1.2.

The remaining problems deal with "perfect" mappings. Recall that a mapping $f : X \to Y$ is *compact* if $f^{-1}\{y\}$ is compact for every $y \in Y$. A mapping is *perfect* if it is continuous, closed and compact.

3. Let $f : X \to Y$ be a closed mapping, let $y \in Y$ and let $G \Subset X$ be such that $f^{-1}\{y\} \subset G$. Show that there exists $O \Subset Y$ such that $y \in O$ and $f^{-1}(O) \subset G$.

4. Let $f : X \to Y$ be a perfect mapping.

   (a) Show that if Y is compact, then X is compact.

   (b) Show that if Y is Lindelöf, then X is Lindelöf.

   [Hint: Use Problem 1.]

5. Let $f : X \to Y$ be a perfect mapping.

   (a) Show that the restriction $f \mid S$ is perfect $X \to Y$ for every $S \subset_c X$.

   (b) Show that the restriction $f \mid f^{-1}(E)$ is perfect $X \to E$ for every $E \subset Y$.

6. Show that the projection map $p_X : X \times C \to X$ is perfect when the space C is compact.

7. Let Z be a Hausdorff space, X a subspace of Z, and let $f : X \to Y$ be a perfect mapping. Show that the graph $\Gamma_f = \{(x, f(x)) : x \in X\}$ of f is closed in $Z \times Y$.

8. Show that there exists a perfect mapping from a Tihonov space X to a space Y iff there exists a Hausdorff compactification C of X such that X is homeomorphic with a closed subspace of $C \times Y$.

   [Hint: Use Problems 3 - 5.]

II CONVERGENCE AND COMPACTNESS

In a metric space, convergent sequences (of points of the space) are a very convenient and useful tool in topological considerations. Convergent sequences determine the topology of a metric space X: a set $S \subset X$ is closed iff S contains all limit points of convergent sequences of points of S. This property of a metric space can be stated as follows: every *sequentially closed* subset is closed. Also we have that a mapping $f : X \to Y$ between metric spaces is continuous iff $f(x_n) \to f(x)$ in Y whenever $x_n \to x$ in X.

Unfortunately, sequences are not equally useful in connection with general topological spaces. The following simple examples show that convergent sequences do not necessarily tell much about the topology of a non-metric space.

A sequence $(x_n)_{n=1}^{\infty}$ is said to be *eventually constant* if there exists $m \in \mathbb{N}$ such that $x_k = x_m$ for every $k > m$. Every eventually constant sequence of points in a topological space is convergent; these are considered to be "trivial" convergent sequences. If x is an isolated point of a space X, then x is not the limit of any non-trivial sequence in X. In (i) and (ii) below, we see that also a non-isolated point may have the same property.

Examples **(i)** Let π be the topology on $\mathbb{R}$ in which every point $p \neq 0$ is isolated and the nbhds of 0 are complements (in $\mathbb{R}$) of countable subsets of $\mathbb{R} \setminus \{0\}$. Then 0 is non-isolated, but no non-trivial sequence converges to 0 in $(\mathbb{R}, \pi)$.

(ii) Denote by τ the usual topology of $\mathbb{Q}$. Let π be the topology on $\mathbb{Q}$ in which every point $p \neq 0$ is isolated and the point 0 has a nbhd base by sets of the form $\{x \in \mathbb{Q} : |x| < r\} \setminus \{x_n : n \in \mathbb{N}\}$, where $r > 0$, $x_k \neq 0$ for each k and $x_n \to 0$ in $(\mathbb{Q}, \tau)$. Note that we have $\tau \subset \pi$.

The point 0 is not isolated in $(\mathbb{Q}, \pi)$. However, 0 is not the limit of any non-trivial sequence. To see this, let $x_n \to 0$ in $(\mathbb{Q}, \pi)$. Since $\tau \subset \pi$, we have that $x_n \to 0$ in $(\mathbb{Q}, \tau)$, and it follows that the set $G = (\mathbb{Q} \setminus \{x_n : n \in \mathbb{N}\}) \cup \{0\}$ is a π-nbhd of 0. Since $x_n \to 0$ in $(\mathbb{Q}, \pi)$, there exists $m \in \mathbb{N}$ such that $\{x_n : n \geq m\} \subset G$. We have that $\{x_n : n \in \mathbb{N}\} \cap G \subset \{0\}$, and it follows that $x_n = 0$ for every $n \geq m$.

(iii) We denote by $[0, \omega_1]$ the space obtained when the ordinal $\omega_1 + 1$ is equipped with its usual order topology (which has a base $\{\{0\}\} \cup \{(\beta, \alpha] : 0 \leq \beta < \alpha \leq \omega_1\}$). The subset $\omega_1 = [0, \omega_1) = [0, \omega_1] \setminus \{\omega_1\}$ is not closed in $[0, \omega_1]$, but it is sequentially closed. To see this, let $\alpha_n < \omega_1$ for $n = 1, 2, 3, ...$, i.e., let α_n be a countable ordinal for each n. Then

$\beta = \sup_{n\in\mathbb{N}} \alpha_n$ is a countable ordinal, and hence $\beta < \omega_1$. Now $(\beta, \omega_1]$ is a nbhd of the point ω_1 and this nbhd contains no point of the sequence $(\alpha_n)_{n=1}^{\infty}$; as a consequence, the sequence does not converge to ω_1. □

Even for general topological spaces, there are meaningful ways to define convergence. This can be succesfully achieved with "nets" or with "filters". Of these two approaches to general convergence below, we shall consider only filters. For nets, the reader is adviced to consult the chapter on that topic in the e-book written by Aisling McCluskey and Briam McMaster; the book can be found at "http://at.yorku.ca/i/a/a/b/23.ps".

II.1. Filters

Perhaps the technically most convenient generalization for sequences is obtained with so called "filters": here we are not in fact generalizing a sequence (x_n) but rather the family $\{\{x_k : k \geq n\} : n \in \mathbb{N}\}$ consisting of the "tails" of the sequence.

Definition A family $\mathcal{L}$ of sets is a *filterbase*, if $\mathcal{L} \neq \emptyset$ and the following conditions hold:
(i) $\emptyset \notin \mathcal{L}$.
(ii) For all $H, L \in \mathcal{L}$ there exists $K \in \mathcal{L}$ such that $K \subset H \cap L$.
A filterbase $\mathcal{L}$ is a *filterbase of a set S* if $\mathcal{L} \subset \mathcal{P}(S)$. A *filter* of S is a filterbase $\mathcal{L}$ of S s.t.
(iii) $L \in \mathcal{L}$ and $L \subset A \subset S \Longrightarrow A \in \mathcal{L}$.

By a *filterbase of a space (Z, τ)* we mean a filterbase of the set Z.

Remarks (1°) *From (i) and (ii) it follows by induction that a filterbase $\mathcal{L}$ is a centered family, i.e., that $\cap\mathcal{H} \neq \emptyset$ for every finite $\mathcal{H} \subset \mathcal{F}$.*
(2°) *For a filter $\mathcal{F}$, we can state (ii) in a stronger form :*
(ii)$_f$ *For all $H, L \in \mathcal{F}$, we have that $H \cap L \in \mathcal{F}$.*

We say that a filterbase $\mathcal{L}$ is *free* if $\bigcap \mathcal{L} = \emptyset$. If $\bigcap \mathcal{L} \neq \emptyset$, then we say that $\mathcal{L}$ is *fixed.*

1 Examples **(a)** If $A \subset S$ and $A \neq \emptyset$, then $\{A\}$ is a filterbase of S.
(b) The family $\big\{\{x_k : k \geq n\} : n \in \mathbb{N}\big\}$ of "tails" of a sequence (x_n) is a filterbase. This filterbase is free provided that $x_i \neq x_j$ whenever $i \neq j$.
(c) The *fixed filter of a set S determined by $s \in S$* is the family $\mathcal{K}_s = \{A \subset S : s \in A\}$.
(d) The *neighbourhood filter of a point $x \in X$* is the family $\eta_x = \{N \subset X : x \in \operatorname{Int} N\}$.

1 Lemma *A family $\mathcal{L}$ of subsets of S is a filterbase iff the family*

$$\mathcal{F} = \{A \subset S : \text{ there exists } L \in \mathcal{L} \text{ such that } L \subset A\}$$

is a filter of S.

Proof. *Necessity.* Assume that $\mathcal{L}$ is a filterbase. We have that $\mathcal{L} \subset \mathcal{F}$ and hence that $\mathcal{F} \neq \emptyset$. For every $F \in \mathcal{F}$, the set F contains some set of the filterbase $\mathcal{L}$ and it follows that $F \neq \emptyset$. Let $F, H \in \mathcal{F}$. Then there exist $L, N \in \mathcal{L}$ such that $L \subset F$ and $N \subset H$. Since $\mathcal{L}$ is a filterbase, there exists $K \in \mathcal{L}$ such that $K \subset L \cap N$. We now have that $K \subset F \cap H$, and hence that $F \cap H \in \mathcal{F}$. The definition of $\mathcal{F}$ shows directly that we have $A \in \mathcal{F}$ whenever $F \subset A \subset S$. By the foregoing, the family $\mathcal{F}$ is a filter of S.
Sufficiency. Assume that $\mathcal{F}$ is a filter of S. Then $\mathcal{F} \neq \emptyset$ and hence there exists a set $F \in \mathcal{F}$; since F contains some set from $\mathcal{L}$, we have that $\mathcal{L} \neq \emptyset$. Since $\mathcal{L} \subset \mathcal{F}$, we have that $\emptyset \notin \mathcal{L}$. If $L, K \in \mathcal{L}$, then $L, K \in \mathcal{F}$ and hence $L \cap K \in \mathcal{F}$; by the definition of $\mathcal{F}$, there exists $H \in \mathcal{L}$ such that $H \subset L \cap K$. We have shown that $\mathcal{L}$ is a filterbase. □

The family $\mathcal{F}$ above is called the filter of S *generated* by the filterbase $\mathcal{L}$.

Examples **(a)** The family of tails of the sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ generates the free filter $\mathcal{F} = \{A \subset \mathbb{R} : \exists \text{such } n \in \mathbb{N} \text{ that } \frac{1}{k} \in A \text{ for every } k \geq n\}$ of $\mathbb{R}$.
(b) Let $s \in S$. The family $\{\{s\}\}$ generates the fixed filter $\mathcal{K}_s$ of S.
(c) A *neighbourhood base* of a point x of a space X is a family of subsets of X which generates the neighbourhood filter η_x of x. Neighbourhood bases are fixed filterbases.

2 Lemma *Let $\mathcal{L}$ and $\mathcal{K}$ be filterbases. Then the family $\mathcal{L} \wedge \mathcal{K} = \{L \cap K : L \in \mathcal{L}$ ja $K \in \mathcal{K}\}$ is a filterbase iff $\emptyset \notin \mathcal{L} \wedge \mathcal{K}$.*

Proof. Exercise. □

We shall now define the notion of convergence for filters of topological spaces.

Definition Let $\mathcal{L}$ be a filterbase of a space X and let $x \in X$. We say that x is a *cluster point* of $\mathcal{L}$ if we have that $x \in \overline{L}$ for every $L \in \mathcal{L}$. We say that $\mathcal{L}$ *converges* to x, and we write $\mathcal{L} \to x$, provided that for every $V \in \eta_x$, there exists $L \in \mathcal{L}$ such that $L \subset V$; then we also say that x is a *limit point* of $\mathcal{L}$.

Remarks (1°) *If $\mathcal{F}$ is the filter on a space X generated by the filterbase $\mathcal{L}$ and $x \in X$, then $\mathcal{F} \to x$ iff $\mathcal{L} \to x$.*

(2°) *For a filter $\mathcal{F}$ of a space X, we can characterize convergence $\mathcal{F} \to x$ by a simpler condition: $\eta_x \subset \mathcal{F}$.*

Examples **(a)** If $(x_n)_{n\in\mathbb{N}}$ is a sequence of points in a space X and $x \in X$, then $x_n \to x$ iff we have $\mathcal{L} \to x$ for the filterbase $\mathcal{L} = \big\{\{x_k : k \geq n\} : n \in \mathbb{N}\big\}$. Moreover, x is a cluster point of the sequence $(x_n)_{n\in\mathbb{N}}$ iff x is a cluster point of the filterbase $\mathcal{L}$.

(b) For every $x \in X$, we have that $\mathcal{K}_x \to x$ and $\eta_x \to x$.

3 Lemma *A point x of a space X is a cluster point of a filterbase $\mathcal{L}$ of X iff there exists a filterbase $\mathcal{N}$ of X such that $\mathcal{L} \subset \mathcal{N}$ and $\mathcal{N} \to x$.*

Proof. *Necessity.* Assume that x is a cluster point of $\mathcal{L}$. For every $L \in \mathcal{L}$, we have that $x \in \overline{L}$ and hence we have that $V \cap L \neq \emptyset$ for every $V \in \eta_x$. By Lemma 2, it follows that the family $\mathcal{N} = \mathcal{L} \wedge \eta_x$ is a filterbase. Moreover, we have that $\mathcal{L} \subset \mathcal{N}$ and $\mathcal{N} \to x$.

Sufficiency. Assume that we have $\mathcal{L} \subset \mathcal{N}$ and $\mathcal{N} \to x$ for a filterbase $\mathcal{N}$ of X. To show that x is a cluster point of $\mathcal{L}$, assume on the contrary that there exists $L \in \mathcal{L}$ such that $x \notin \overline{L}$. Then $X \setminus L$ is a nbhd of x and it follows, since $\mathcal{N} \to x$, that there exists $N \in \mathcal{N}$ such that $N \subset X \setminus L$. This, however, is a contradiction, since $\mathcal{N}$ is a filterbase and $N, L \in \mathcal{N}$. □

In particular, the lemma shows that a limit point of a filterbase is a cluster point.

The next results show that filter convergence determines the topology of a space.

4 Proposition *TFAE for a subset A and a point x of a space X:*

A. $x \in \overline{A}$.

B. *The point x is a cluster point of some filterbase of A.*

C. *The family $\mathcal{L} = \{V \cap A : V \in \eta_x\}$ is a filterbase and $\mathcal{L} \to x$.*

Proof. *A⇒C***:** Assume that $x \in \overline{A}$. Then $V \cap A \neq \emptyset$ for each $V \in \eta_x$, and Lemma 2C shows that the family $\mathcal{L} = \eta_x \wedge \{A\}$ is a filterbase. Clearly $\mathcal{L} \to x$.

*C⇒B***:** By Lemma 3.

*B⇒A***:** Assume that x is a cluster point of the filterbase $\mathcal{K}$ of the set A. Let $K \in \mathcal{K}$. Then $K \subset A$ and $x \in \overline{K}$ and hence $x \in \overline{A}$. □

Note that if X is T_1 and $x \in \overline{A}$, then the filterbase $\mathcal{L}$ in C above is fixed iff $x \in A$.

By the above result, we see that a subset A of a space X is closed iff A contains all limit points of its filters. This means that the topology of X is uniquely determined by filter convergence in X.

By considering indiscrete topologies $\{X, \emptyset\}$, we see that one filterbase can converge to several points. We now characterize those spaces where this cannot happen.

5 Theorem *A topological space X is Hausdorff iff every filterbase of X has at most one limit point.*

Proof. *Necessity.* Assume that X is Hausdorff. Let $\mathcal{L}$ be a filterbase of X such that $\mathcal{L} \to x$ and $\mathcal{L} \to y$ in X. We show that $x = y$. Assume on the contrary that $x \neq y$. Sincea X is Hausdorff, there are nbhds $U \in \eta_x$ and $V \in \eta_y$ such that $U \cap V = \emptyset$. Since $\mathcal{L} \to x$ and $\mathcal{L} \to y$, there exist $H, K \in \mathcal{L}$ such that $H \subset U$ and $K \subset V$. Further, there exists $L \in \mathcal{L}$ such that $L \subset H \cap K$. Now we have that $L \subset H \cap K \subset U \cap V = \emptyset$, in other words, that $L = \emptyset$. This is a contradiction, since $\mathcal{L}$ is a filterbase.
Sufficiency. Assume that every filterbase of X has at most one limit point. We show that X is Hausdorff. Let x and y be points of X such that we have $U \cap V \neq \emptyset$ for all $U \in \eta_x$ and $V \in \eta_y$. We show that $x = y$. By Lemma 2, the family $\mathcal{L} = \{U \cap V : U \in \eta_x \text{ and } V \in \eta_y\}$ is a filterbase of X. Directly from the definition of $\mathcal{L}$ it follows that $\mathcal{L} \to x$ and $\mathcal{L} \to y$; our assumption now shows that $x = y$. □

Next we show that continuity of mappings between spaces can be very naturally characterized in terms of filter convergence. The intuitive idea of continuity of a map $f : X \to Y$ is that f does not "tear" the space X which it is mapping, and this is equivalent with the idea that for points which are "near to each other" in X, the images are "near to each other" in Y. We can measure "nearness" for example with nbhds: points in a "small" nbhd of a point are thought to be "close" to the point. This idea leads to following definition of continuity: f is continuous at a point $x \in X$ provided that for every $W \in \eta_{f(x)}(Y)$, there exists $V \in \eta_x(X)$ such that $f(V) \subset W$. This condition can be stated more simply as follows: f is continuous at a point $x \in X$ provided that $f^{-1}(W) \in \eta_x(X)$ for every $W \in \eta_{f(x)}(Y)$. The mapping f is continuous if f is continuous at each point of X. Using the second formulation of continuity at a point, we easily get the following result: f is continuous iff $f^{-1}(G) \Subset X$ for every $G \Subset Y$.

In a metric space, the idea of "nearness" can be captured by convergent sequences: if

$x_n \to x$, then the points x_n are "getting nearer and nearer" to the point x. This translates to the following notion of continuity: $f : X \to Y$ is continuous at $x \in X$ provided that $f(x_n) \to f(x)$ in Y whenever $x_n \to x$. This condition is, in fact, equivalent with the earlier conditions *for metric spaces* X and Y. For general spaces, we cannot characterize continuity in terms of convergent sequences, but we obtain a similar characterization using convergence of filters.

Let $f : X \to Y$. For every family $\mathcal{H}$ of subsets of X, we set $f(\mathcal{H}) = \{f(H) : H \in \mathcal{H}\}$. It is easy to see that $f(\mathcal{H})$ is a filterbase if $\mathcal{H}$ is a filterbase.

6 Theorem *TFAE for a mapping $f : X \to Y$ and a point $x \in X$:*
(1) *f is continuous at x.*
(2) *For every $A \subset X$, if $x \in \overline{A}$, then $f(x) \in \overline{f(A)}$.*
(3) *If x is a cluster point of a filterbase $\mathcal{L}$ of X, then $f(x)$ is a cluster point of $f(\mathcal{L})$.*
(4) *For every filterbase $\mathcal{L}$ of X, if $\mathcal{L} \to x$, then $f(\mathcal{L}) \to f(x)$.*

Proof. (1)⇒(4): Assume that f is continuous at x and the filterbase $\mathcal{L}$ converges to x. To show that $f(\mathcal{L}) \to f(x)$, let $W \in \eta_{f(x)}(Y)$. Since f is continuous at x, we have that $f^{-1}(W) \in \eta_x(X)$ and it follows, since $\mathcal{L} \to x$, that there exists $L \in \mathcal{L}$ such that $L \subset f^{-1}(W)$. Now $f(L) \in f(\mathcal{L})$ and $f(L) \subset W$. We have shown that $f(\mathcal{L}) \to f(x)$.
(4)⇒(3): This follows by Lemma 3.
(3)⇒(2): Assume that (3) holds and $A \subset X$ is such that $x \in \overline{A}$. The family $\mathcal{L} = \{A\}$ is a filterbase of X and x is a cluster point of $\mathcal{L}$. Since (3) holds, we have that $f(x)$ is a cluster point of $f(\mathcal{L})$, in other words, we have that $f(x) \in \overline{f(A)}$.
(2)⇒(1): Assume that (2) holds. To show that f is continuous at x, assume on the contrary that there exists $W \in \eta_{f(x)}(Y)$ such that $f^{-1}(W) \notin \eta_x(X)$. Then we have that $x \in \overline{X \setminus f^{-1}(W)}$, and it follows by our assumption that $f(x) \in \overline{f(X \setminus f^{-1}(W))}$; this, however, is a contradiction, since $f(X \setminus f^{-1}(W)) \subset Y \setminus W$ and $W \in \eta_{f(x)}(Y)$. □

Next we characterize filter convergence in product spaces; the result explains why the product topology is often called the "topology of pointwise convergence".

7 Theorem *Let $\mathcal{L}$ be a filterbase in a product space $\prod_{i \in I} X_i$, and let $(x_i)_{i \in I} \in X$.*
A. *If $(x_i)_{i \in I}$ is a cluster point of $\mathcal{L}$, then x_i is a cluster point of $p_i(\mathcal{L})$ for each i.*
B. *$\mathcal{L} \to (x_i)_{i \in I}$ iff $p_i(\mathcal{L}) \to x_i$ for every $i \in I$.*